Problem 4.14

(a) Using Equation 4.88, work out the first four Laguerre polynomials.

(b) Using Equations 4.86, 4.87, and 4.88, find $v(\rho)$, for the case $n = 5$, $\ell = 2$.

(c) Find $v(\rho)$ again (for the case $n = 5$, $\ell = 2$), but this time get it from the recursion formula (Equation 4.76).

Solution

Part (a)

Equation 4.88 is on page 150 and gives the formula for the qth Laguerre polynomial.

$$L_q(x) = \frac{e^x}{q!}\left(\frac{d}{dx}\right)^q (e^{-x}x^q) \tag{4.88}$$

Set $q = 0$.

$$\begin{aligned} L_0(x) &= \frac{e^x}{0!}\left(\frac{d}{dx}\right)^0 (e^{-x}x^0) \\ &= \frac{e^x}{1}(1)(e^{-x}) \\ &= 1 \end{aligned}$$

Set $q = 1$.

$$\begin{aligned} L_1(x) &= \frac{e^x}{1!}\left(\frac{d}{dx}\right)^1 (e^{-x}x^1) \\ &= \frac{e^x}{1}\frac{d}{dx}(xe^{-x}) \\ &= e^x[e^{-x} + xe^{-x}\cdot(-1)] \\ &= 1 - x \end{aligned}$$

Set $q = 2$.

$$\begin{aligned} L_2(x) &= \frac{e^x}{2!}\left(\frac{d}{dx}\right)^2 (e^{-x}x^2) \\ &= \frac{e^x}{2}\frac{d^2}{dx^2}(x^2e^{-x}) \\ &= \frac{e^x}{2}\frac{d}{dx}(2xe^{-x} - x^2e^{-x}) \\ &= \frac{e^x}{2}(2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}) \\ &= \frac{1}{2}(2 - 4x + x^2) \end{aligned}$$

Set $q = 3$.

$$\begin{aligned}
L_3(x) &= \frac{e^x}{3!}\left(\frac{d}{dx}\right)^3 (e^{-x}x^3) \\
&= \frac{e^x}{6}\frac{d^3}{dx^3}(x^3e^{-x}) \\
&= \frac{e^x}{6}\frac{d^2}{dx^2}(3x^2e^{-x} - x^3e^{-x}) \\
&= \frac{e^x}{6}\frac{d}{dx}(6xe^{-x} - 3x^2e^{-x} - 3x^2e^{-x} + x^3e^{-x}) \\
&= \frac{e^x}{6}\frac{d}{dx}(6xe^{-x} - 6x^2e^{-x} + x^3e^{-x}) \\
&= \frac{e^x}{6}(6e^{-x} - 6xe^{-x} - 12xe^{-x} + 6x^2e^{-x} + 3x^2e^{-x} - x^3e^{-x}) \\
&= \frac{e^x}{6}(6e^{-x} - 18xe^{-x} + 9x^2e^{-x} - x^3e^{-x}) \\
&= \frac{1}{6}(6 - 18x + 9x^2 - x^3)
\end{aligned}$$

Set $q = 4$.

$$\begin{aligned}
L_4(x) &= \frac{e^x}{4!}\left(\frac{d}{dx}\right)^4 (e^{-x}x^4) \\
&= \frac{e^x}{24}\frac{d^4}{dx^4}(x^4e^{-x}) \\
&= \frac{e^x}{24}\frac{d^3}{dx^3}(4x^3e^{-x} - x^4e^{-x}) \\
&= \frac{e^x}{24}\frac{d^2}{dx^2}(12x^2e^{-x} - 4x^3e^{-x} - 4x^3e^{-x} + x^4e^{-x}) \\
&= \frac{e^x}{24}\frac{d^2}{dx^2}(12x^2e^{-x} - 8x^3e^{-x} + x^4e^{-x}) \\
&= \frac{e^x}{24}\frac{d}{dx}(24xe^{-x} - 12x^2e^{-x} - 24x^2e^{-x} + 8x^3e^{-x} + 4x^3e^{-x} - x^4e^{-x}) \\
&= \frac{e^x}{24}\frac{d}{dx}(24xe^{-x} - 36x^2e^{-x} + 12x^3e^{-x} - x^4e^{-x}) \\
&= \frac{e^x}{24}(24e^{-x} - 24xe^{-x} - 72xe^{-x} + 36x^2e^{-x} + 36x^2e^{-x} - 12x^3e^{-x} - 4x^3e^{-x} + x^4e^{-x}) \\
&= \frac{e^x}{24}(24e^{-x} - 96xe^{-x} + 72x^2e^{-x} - 16x^3e^{-x} + x^4e^{-x}) \\
&= \frac{1}{24}(24 - 96x + 72x^2 - 16x^3 + x^4)
\end{aligned}$$

Part (b)

Equations 4.86 and 4.87 are on page 149, and Equation 4.88 is on page 150.

$$v(\rho) = L_{n-\ell-1}^{2\ell+1}(2\rho) \tag{4.86}$$

$$L_q^p(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_{p+q}(x) \tag{4.87}$$

$$L_q(x) = \frac{e^x}{q!}\left(\frac{d}{dx}\right)^q (e^{-x}x^q) \tag{4.88}$$

Evaluate $v(\rho)$ for $n = 5$ and $\ell = 2$.

$$\begin{aligned}
v(\rho) &= L_{5-2-1}^{2(2)+1}(2\rho) \\
&= L_2^5(2\rho) \\
&= (-1)^5 \left(\frac{d}{dx}\right)^5 L_{2+5}(x)\bigg|_{x=2\rho} \\
&= -\frac{d^5}{dx^5}[L_7(x)]\bigg|_{x=2\rho} \\
&= -\frac{d^5}{dx^5}\left[\frac{e^x}{7!}\left(\frac{d}{dx}\right)^7 (e^{-x}x^7)\right]\bigg|_{x=2\rho} \\
&= -\frac{d^5}{dx^5}\left[\frac{e^x}{5040}\frac{d^7}{dx^7}(x^7e^{-x})\right]\bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^6}{dx^6}(7x^6e^{-x} - x^7e^{-x})\right]\bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^5}{dx^5}(42x^5e^{-x} - 7x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x})\right]\bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^5}{dx^5}(42x^5e^{-x} - 14x^6e^{-x} + x^7e^{-x})\right]\bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^4}{dx^4}(210x^4e^{-x} - 42x^5e^{-x} - 84x^5e^{-x} + 14x^6e^{-x} + 7x^6e^{-x} - x^7e^{-x})\right]\bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^4}{dx^4}(210x^4e^{-x} - 126x^5e^{-x} + 21x^6e^{-x} - x^7e^{-x})\right]\bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\bigg[e^x\frac{d^3}{dx^3}(840x^3e^{-x} - 210x^4e^{-x} - 630x^4e^{-x} + 126x^5e^{-x} + 126x^5e^{-x} \\
&\qquad - 21x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x})\bigg]\bigg|_{x=2\rho}
\end{aligned}$$

Finish evaluating derivatives and then simplify the result.

$$
\begin{aligned}
v(\rho) &= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^3}{dx^3}(840x^3e^{-x} - 840x^4e^{-x} + 252x^5e^{-x} - 28x^6e^{-x} + x^7e^{-x})\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^2}{dx^2}(2520x^2e^{-x} - 840x^3e^{-x} - 3360x^3e^{-x} + 840x^4e^{-x} + 1260x^4e^{-x} - 252x^5e^{-x}\right. \\
&\qquad \left. - 168x^5e^{-x} + 28x^6e^{-x} + 7x^6e^{-x} - x^7e^{-x})\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d^2}{dx^2}(2520x^2e^{-x} - 4200x^3e^{-x} + 2100x^4e^{-x} - 420x^5e^{-x} + 35x^6e^{-x} - x^7e^{-x})\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d}{dx}(5040xe^{-x} - 2520x^2e^{-x} - 12600x^2e^{-x} + 4200x^3e^{-x} + 8400x^3e^{-x} - 2100x^4e^{-x}\right. \\
&\qquad \left. - 2100x^4e^{-x} + 420x^5e^{-x} + 210x^5e^{-x} - 35x^6e^{-x} - 7x^6e^{-x} + x^7e^{-x})\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x\frac{d}{dx}(5040xe^{-x} - 15120x^2e^{-x} + 12600x^3e^{-x}\right. \\
&\qquad \left. - 4200x^4e^{-x} + 630x^5e^{-x} - 42x^6e^{-x} + x^7e^{-x})\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x(5040e^{-x} - 5040xe^{-x} - 30240xe^{-x} + 15120x^2e^{-x} + 37800x^2e^{-x} - 12600x^3e^{-x}\right. \\
&\qquad - 16800x^3e^{-x} + 4200x^4e^{-x} + 3150x^4e^{-x} - 630x^5e^{-x} - 252x^5e^{-x} \\
&\qquad \left. + 42x^6e^{-x} + 7x^6e^{-x} - x^7e^{-x})\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}\left[e^x(5040e^{-x} - 35280xe^{-x} + 52920x^2e^{-x} - 29400x^3e^{-x}\right. \\
&\qquad \left. + 7350x^4e^{-x} - 882x^5e^{-x} + 49x^6e^{-x} - x^7e^{-x}\right]\Bigg|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}(5040 - 35280x + 52920x^2 - 29400x^3 + 7350x^4 - 882x^5 + 49x^6 - x^7)\Big|_{x=2\rho} \\
&= -\frac{1}{5040}\frac{d^5}{dx^5}(-882x^5 + 49x^6 - x^7)\Big|_{x=2\rho} \\
&= -\frac{1}{5040}\left(-882\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot x^{5-5} + 49\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot x^{6-5} - 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot x^{7-5}\right)\Big|_{x=2\rho} \\
&= -\frac{1}{5040}\left(-105840 + 35280x - 2520x^2\right)\Big|_{x=2\rho}
\end{aligned}
$$

Therefore,

$$\begin{aligned} v(\rho) &= -\frac{1}{5040}\left[-105840 + 35280(2\rho) - 2520(2\rho)^2\right] \\ &= -\frac{1}{5040}\left[-105840 + 35280(2\rho) - 2520(4\rho^2)\right] \\ &= 21 - 14\rho + 2\rho^2. \end{aligned}$$

Part (c)

$v(\rho)$ is defined in Equation 4.62 on page 145 as a power series solution to the transformed radial equation for the hydrogen atom.

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j \tag{4.62}$$

The coefficients c_j are determined from the recursion relation in Equation 4.76 on page 147.

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)} c_j, \quad j = 0, 1, 2, \ldots \tag{4.76}$$

$v(\rho)$ is expected to be a polynomial of degree $n - \ell - 1$ with a normalization constant. If $n = 5$ and $\ell = 2$, then

$$c_{j+1} = \frac{2(j+2+1-5)}{(j+1)[j+2(2)+2]} c_j = \frac{2(j-2)}{(j+1)(j+6)} c_j.$$

Plug in values for j to get the coefficients.

$$\begin{aligned} j = 0: &\quad c_1 = \frac{2(-2)}{(1)(6)} c_0 = -\frac{2}{3} c_0 \\ j = 1: &\quad c_2 = \frac{2(-1)}{(2)(7)} c_1 = -\frac{1}{7}\left(-\frac{2}{3} c_0\right) = \frac{2}{21} c_0 \\ j = 2: &\quad c_3 = \frac{2(0)}{(3)(8)} c_2 = 0 \\ j = 3: &\quad c_4 = \frac{2(1)}{(4)(9)} c_3 = 0 \\ j = 4: &\quad c_5 = \frac{2(2)}{(5)(10)} c_4 = 0 \\ &\quad\vdots \end{aligned}$$

Therefore,

$$\begin{aligned} v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j = c_0 + c_1\rho + c_2\rho^2 &= c_0 + \left(-\frac{2}{3} c_0\right)\rho + \left(\frac{2}{21} c_0\right)\rho^2 \\ &= c_0\left(1 - \frac{2}{3}\rho + \frac{2}{21}\rho^2\right). \end{aligned}$$